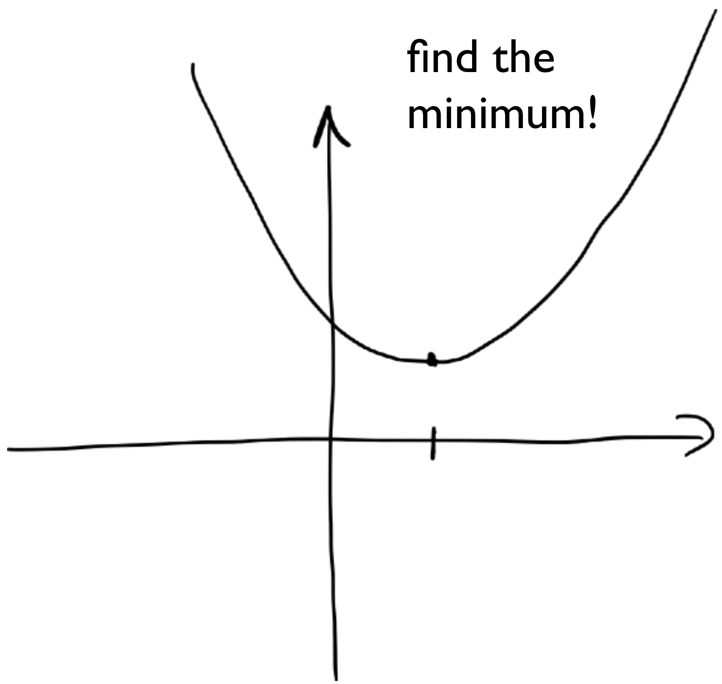
Optimization models

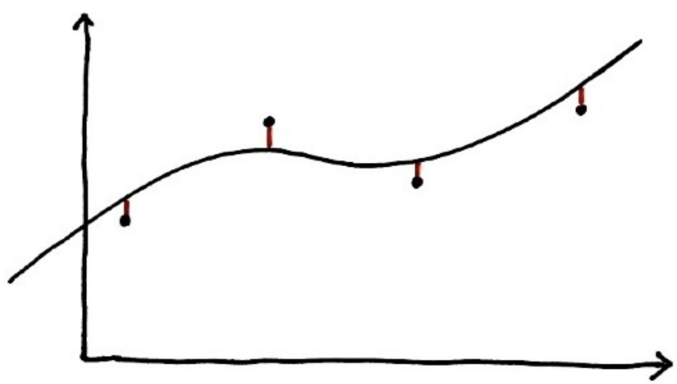
Unconstrained optimization



You have probably seen this!

But what is an optimization problem more generally?

Least squares method



Find best curve

Minimize quadratic error

Not just for straight lines!

What is an optimization problem?

Example: shortest path problem

Minimize

the length

over

all paths from a to b in a graph

Minimize

the objective function

over

the set of feasible solutions

How should a can be designed?

What is the question?

What type of model seems appropriate?

Formulating the model

formulate the problem mathematically

How minimize area for given volume?

some variables

V volume

A area

r radius

h height

some equations

$$V = \pi r^2 h$$

$$A = 2\pi r^2 + 2\pi r h$$

the optimization model

$$\min \; 2\pi r^2 + 2\pi r h$$

Objective function

when

$$\pi r^2 h = 1$$

$$r \geq 0$$

$$h \geq 0$$

Constraints describe the feasible set

Note: only r and h appear as variables in the optimization problem!
The solution to the problem are values for r and h
Note that both the objective function and the constant are nonlinear!

So we have seen different ways to formulate an optimization problem

With words

Common for problems over discrete structures, e.g. shortest path problem, minimum spanning tree,…

With variables and equations

in this module

This is called *mathematical programming.*

Ordinary system of equations

$$4x_1 + 3x_2 + 5x_3 = 4$$

$$8x_1 + x_2 + 2x_3 = 0$$

$$3x_1 + 5x_3 = -5$$

only equalities

same number of equations as variables

usually one solution

In an optimization problem…

equalities and
inequalities!

$$4x_1 + 3x_2 + 5x_3 = 4$$

$$8x_1 + 3x_2 \geqslant 2$$

any number of
equations and
inequalites!

usually many
solutions…

$$\min\ 2x_1 + 3x_2 + x_3$$

subject to

$$4x_1 + 3x_2 + 5x_3 = 4$$

$$8x_1 + 3x_2 \geq 2$$

so we use an objective function to tell which of these solutions we desire! (minimize or maximize)

<u>much</u> more flexible and powerful!

NOTE just one objective function!

So constrained optimization combines…

system of equations
or other constraints

unconstrained
optimization

constrained optimization

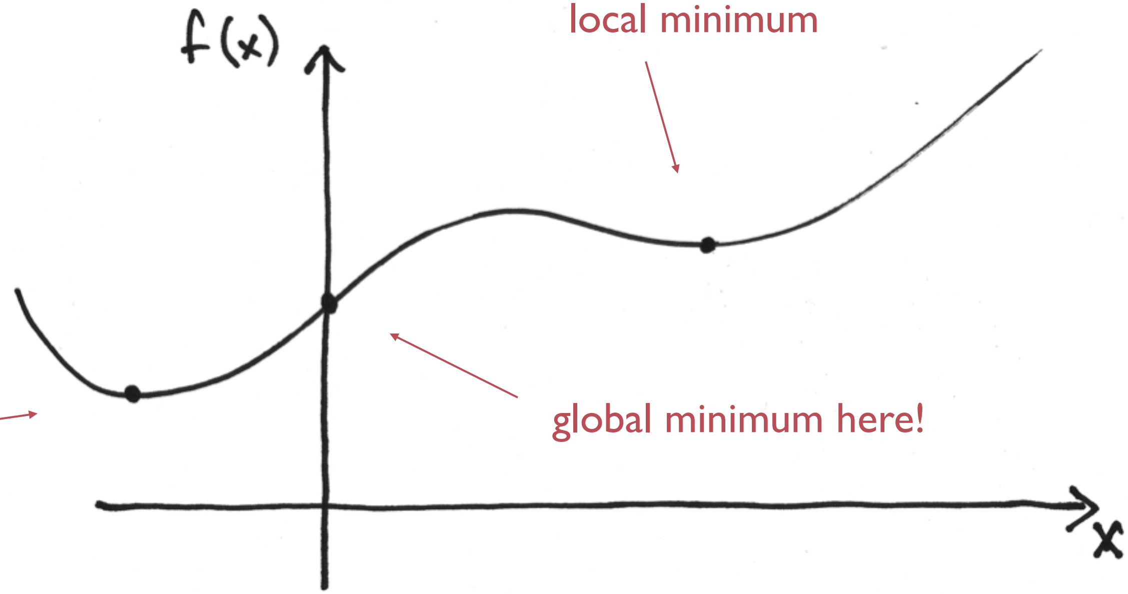
A powerful combination! But usually more difficult to solve than each part separately. Fortunately there are many nice algorithms!

Why is it more complicated with constraints?

a single constraint

$$\min f(x)$$
$$\text{subject to } x \geq 0$$

local minimum



global minimum here!

not feasible!

Now imagine many variables and constraints…

Some variations

variables can be continuous, integer, or binary

$$\min\ 2x_1 + 3x_2 + x_3$$

subject to

$$4x_1 + 3x_2 + 5x_3 = 4$$

$$8x_1 + 3x_2 \geqslant 2$$

equations, inequalities and the objective function can be linear or non-linear

<u>Linear programming:</u>

- continuous variables
- equation/inequalities and objective function linear

important and easily solved by computers for thousands or even millions of variables!

how model an optimization problem?

Always first!

Try to intuitively understand the problem, understand what the question is, and in principle what the solution is!

Formulating a mathematical programming problem

A. Define variables.

B. Define constraints (equations and inequalities) with the variables.

C. Define the objective function.

B and C can be done in any order depending on what is most convenient for the problem at hand.

This is the modelling step!

A. Defining the variables

How can a solution be represented?

Different kinds of variables:

- continuous
- continuous nonnegative
- integer (categorical…)
- binary

Distinguish between:

- variables and constants!
- math variables and programming variables!

If you cannot clearly see what the variables are, it can be useful to give mathematical names to *all mathematical entities that you can think of,* even if they don't end up directly as variables in the optimization problem. This was the case e.g. for V and A in the can problem. You may also wish to define various *symbolic constants* that help to describe the problem.

B. Defining constraints

Do I know any equations or inequalities that must hold for any correct solution?

Just like setting up a system of equations but more flexible:

- You can have fewer equations than variables!
- You can use inequalities!
- You can have an objective function that determines which of all possible solutions you are interested in!

If you cannot clearly see what the constraints are, can be useful to write down *all equations and inequalities that you can think of,* even if they don't end up directly as constraints in the optimization problem. This was the case e.g. for the equations for V and A in the can problem.

C. Define the objective function

How can I calculate the quality of a solution if I know it?

Do you wish to minimize or maximize?

Just one objective function!

Simple assignment

tasks

persons

1	3	5	1
4	5	3	2
7	4	6	9
8	4	7	3

How match tasks and persons to minimize cost?

Simple assignment model

a constant parameter

a variable

$$\min\ c_{11}x_{11} + c_{12}x_{12} + \ldots + c_{44}x_{44}$$

subject to

$$\begin{cases} x_{11} + x_{12} + x_{13} + x_{14} = 1 \\ \vdots \end{cases}$$

$$\begin{cases} x_{11} + x_{21} + x_{31} + x_{41} = 1 \\ \vdots \end{cases}$$

$$x_{ij} \text{ binary}$$

a binary integer linear program (ILP)

A common simplification

$$x_{ij} \text{ binary} \implies 0 \leq x_{ij} \leq 1$$

ILP LP !

Easier to solve but can cause other difficulties

Always use linear constraints and linear objective function when possible, since algorithms are <u>much</u> more efficient for this case.

Many variables are ok!

Modelling hints

You don't have to solve the problem when you set it up!

Often it is not so clear what kind of problem you have. Just start writing equations and see what you get along the way!

Some problems are best solved with a more intuitive approach!