4 Mostly optimization

Depending on your background, some of these exercises may be easy, while others may be more difficult. We simply ask you to do your best with these problems, within the limits of the time allocated for the course.

Read **these reminders** before you begin!

(keeping track)

(WHAT IS THE REVENUE?)

Consider this data for sales and pricing of chocolate bars:

year	quantity	unit price
2014	10300	15 kr
2015	8100	17 kr
2016	7400	18 kr

a) Assuming that this data gives an indication of the price sensitivity of chocolate bar sales, how can you estimate a) for which price revenue is maximized, b) for which price profit is maximized, if we have a production cost of 3 kr per bar (ignoring fixed costs). Perform any calculations as far as possible.

b) Even though you do not know much about the specific situation, hypothesize other possible reasons for the decline in chocolate bar sales.

(investigating the abstract)

(LEAST SQUARES METHOD)

Fitting a curve graphically can work fine, but there is a limitation in that you then do not have a precise criterion for what is a good fit. The most common criterion in mathematics is the **least squares method** (look at any other websites if you like), which minimizes the sum of the squares of the errors between the points and the fitted function. Note how the curve fitting then becomes a well defined optimization problem!

Mathematica uses the least squares criterion in the function `Fit`, which finds the best linear combination (=weighted sum) of a set of base functions to minimize the quadratic error. For the **(EMPIRICAL CURVE FITTING)** from the last module, try this out by automatically fitting a quadratic polynomial to the points:

```
data = {{88.0, 57.9}, {224.7, 108.2}, {365.3,
149.6}, {687.0,
    228.07}, {4332, 778.434}, {10760, 1428.74},
{30684,
    2839.08}, {60188, 4490.8}, {90467, 5879.13}};
f = Fit[data, {1, x, x^2}, x]
```

For the quadratic polynomial we use the *base functions* 1,x and x^2, and the function will try to fit a *linear combination* (weighted average) of these functions. So this means that we ask to find a function of the form ax^2+bx+c. Then you can plot with:

```
p1 = ListPlot[data];
p2 = Plot[f, {x, 0, 150000}, PlotStyle -> Red];
Show[p1, p2, PlotRange -> {{0, 150000}, {0,
8000}}]
```

a) As the answer to the question, show the function and the plot.
b) Why do you think it may be good to minimize the sum of square errors. Why not simply the sum of the errors for each point?

(SIMPLE ASSIGNMENT PROBLEM)

Take a careful look at how we modelled this problem in the lecture. Try Mathematica for this problem, formulated as a linear programming problem (no integer constraints). Make sure that you understand how the problem is formulated in Mathematica. See what solution you get. Is it fractional or did you get an integer solution?

```
NMinimize[
 {1 x11 + 3 x12 + 5 x13 + 1 x14 + 4 x21 + 5 x22 +
3 x23 + 2 x24 +
   7 x31 + 4 x32 + 6 x33 + 9 x34 + 8 x41 + 4 x42
+ 7 x43 + 3 x44,
  x11 + x12 + x13 + x14 == 1,
  x21 + x22 + x23 + x24 == 1,
  x31 + x32 + x33 + x34 == 1,
  x41 + x42 + x43 + x44 == 1,
  x11 + x21 + x31 + x41 == 1,
  x12 + x22 + x32 + x42 == 1,
  x13 + x23 + x33 + x43 == 1,
  x14 + x24 + x34 + x44 == 1,
  0 <= x11 <= 1, 0 <= x12 <= 1, 0 <= x13 <= 1, 0
<= x14 <= 1,
  0 <= x21 <= 1, 0 <= x22 <= 1, 0 <= x23 <= 1, 0
<= x24 <= 1,
  0 <= x31 <= 1, 0 <= x32 <= 1, 0 <= x33 <= 1, 0
<= x34 <= 1,
```

```
  0 <= x41 <= 1, 0 <= x42 <= 1, 0 <= x43 <= 1, 0
<= x44 <= 1},
 {x11, x12, x13, x14, x21, x22, x23, x24, x31,
x32, x33, x34, x41,
  x42, x43, x44}
 ]
```

(WHEN IS AN OPTIMAL SOLUTION GUARANTEED?)

Numerical optimization algorithms often work with some variant of the following iterative approach:

- Make sure to be in a feasible point.
- Take a small step in a direction that will improve the objective function while still staying within the feasible set.
- Repeat until no improvement.

A natural and important question is then if an algorithm of this kind is guaranteed to find the globally optimal solution, or if it is at risk to get stuck in a local optimum.

a) Give examples of optimization problems in one and two variables, where this approach will always work, and examples of problems where it may fail. HINT The two-dimensional problem can be seen as finding a highest or a lowest point in a landscape, within a limited geographical area.

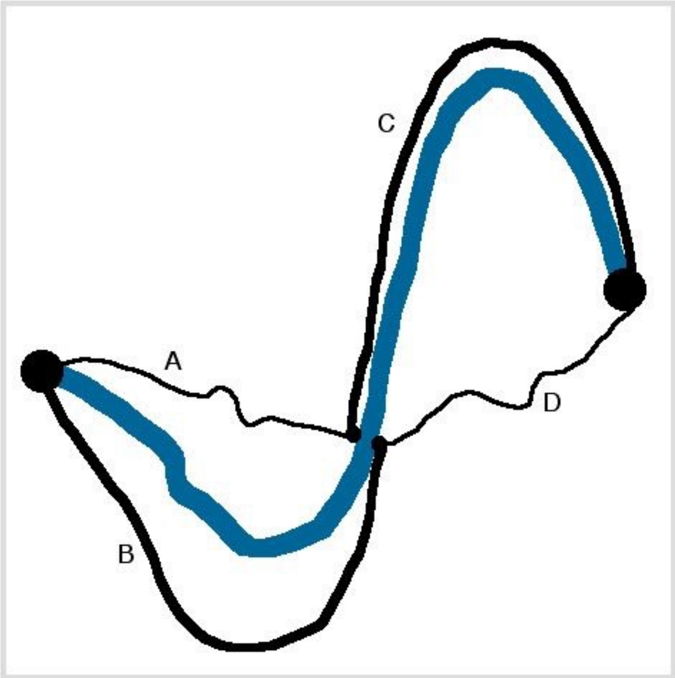
b) (Voluntary) What is the most general case you can think of, where finding the global optimum can be guaranteed? Consider both the character of the objective function and the shape of the feasible set.

(investigating the world)

(BRIDGE PROBLEM)

Consider the road network below. The figure illustrates the roads between two larger cities along a river. Roads B and C are large roads and have a fixed travel time of about 30 minutes independently of the traffic load. The roads A and D are mountain roads and the travel time is estimated to about 10+x minutes where x is the traffic intensity in cars per minute (in one direction). During rush hours the total traffic between the cities in either direction s is about 20 cars per minute.

a) Assuming that every individual tries to optimize his/her own travel time, what will be the travel time during rush hours? Motivate your answer. Hint: consider if all individual decisions of drivers could eventually lead to some equilibrium state with constant traffic flows? (note that this is not a single optimization problem, but a more complex situation with many agents where each agent optimizes for itself)



b) In order to improve traffic flow it is decided to build a bridge over the river between the two small communities. The travel time over the bridge is about 1 minute independently of the traffic volumes we are considering. Again, assuming that every individual tries to optimize his/her own travel time, what will be the new travel time between the large cities? Motivate your answer, discuss the result and draw qualitative conclusions.

(designing)

(OPTIMAL SHAPE OF CAN)

Take a careful look at how we modelled this problem in the lecture. Try Mathematica for this problem and find out what the best shape is! (Note that this is a non-linear optimization problem, which are usually harder to solve, but this is a very small one)

```
FindMinimum[{2 Pi r h + 2 Pi r^2 , Pi r^2 h == 1, r
>= 0, h >= 0}, {r, h}]
```

(EMERGENCY CARE PROBLEM)

The following problem is a so called facility location problem. A city wishes to make a long term study to decide where to best locate emergency care. The city has been partitioned into regions, and it has been decided that an emergency care site can acceptably service regions of the city which are within a driving distance of 8 minutes. The goal is to choose a set of stations at minimum cost. There are seven regions to cover, and six potential sites have been identified.

Distances in minutes between regions and potential sites:

Site #	1	2	3	4	5	6
Region 1	15	3	12	5	17	20
Region 2	12	9	13	16	3	4
Region 3	13	16	9	4	7	11
Region 4	3	7	6	22	5	18
Region 5	4	22	12	5	16	14
Region 6	8	10	5	16	13	5
Region 7	13	10	5	6	13	21

The cost for locating emergency care on the respective sites:

	Cost
Site 1	710 000
Site 2	610 000
Site 3	650 000
Site 4	910 000
Site 5	720 000
Site 6	570 000

a) Model this problem mathematically by defining variables, constraints and an objective function. To get started, you can simply begin to define some variables, write some equations and see what you get along the way. (It is best if you can make the constraints and the objective function linear, since then the problem becomes easier to solve mathematically. For links about linear programming see below. Note that in this step you are not solving the problem, just defining it. Hint: think about what I said in the lecture about how to define the variables.

b) Now try to solve your model by using the Mathematica function `NMinimize`. Try to solve it as a plain linear programming problem with continuous variables (without using any special options of the `NMinimize` function or constraints to say that the variables are integer or binary - as we discussed in the introductory lecture). Describe any difficulties you run into. What conclusions can you draw from the solution you obtain? Then solve the original integer problem eg. by solving variations of the LP multiple times, or by using some integer option in Mathematica, and give the answer.

c) Ask yourself if this is the only way to handle this problem? For example, in this problem we assumed that we should have a maximum number of minutes from each region of the city. Is this the only way to think about this? If you can, elaborate on any idea you might have. Note that while this problem only has a small number of variables and therefore can be solved by brute force combinatorial search, this would be useless for larger problems. For larger problems the more mathematical approach is much more powerful.

(thinking)

(REASONING - PSYCHOLOGICAL TEST)

For each of the four cards below there is a letter on one side and a digit on the other.

F 8 U 3

For these cards it is also claimed that if there is a vowel on one side, there is an odd number on the other side. What is the least number of cards you need to turn to verify this, and which cards do you need to turn?

(mathematical knowledge)

(LINEAR PROGRAMMING)

You can use e.g. the following links to read about linear programming in general:

- https://en.wikipedia.org/wiki/Linear_programming
- linear programming
- integer linear programming

You can also have a look in a book on optimization or operations research.

HINT Focus on understanding the problem of linear programming, rather than trying to learn or understand algorithms for this problem, which is beyond this course.

(LINEAR ALGEBRA - MATRICES AND SYSTEMS OF EQUATIONS)

It is common to write systems of linear equations with matrices and vectors, and knowing this notation is necessary for reading many applied mathematical texts.

Read at https://simple.wikipedia.org/wiki/Matrix_(mathematics)

Also read about systems of linear equations as a linear combination (=weighted sum) of vectors at https://en.wikipedia.org/wiki/System_of_linear_equations (Especially the part on general form - but it is all very worthwhile reading if you do not already know this well).

(finally...)

(MID-COURSE FEEDBACK)

The purpose of this question is to ensure a common understanding of what we are trying to do in this course, and to identify any possible problems. Your answers may influence our feedback and other actions during the rest of the course.

a) Please write what you think is the purpose of this course. If you like, you may also write

a personal answer in terms of how you see the purpose of the course for you.

b) Does the course work well for you? If not, please explain!

c) Any other comments?

d) What is your math background? The main distinction for us is “high school” or “university”, but you are welcome to provide additional information if you wish. Please also give your main field of study in your BSc.

(SELF-CHECK)

- Have you answered all questions to the best of your ability?
- Is the required information on the front page, file name correct etc.?
- Anything else you can easily check?

If you pass the self-check, simply write "Self-check passed!”. Otherwise, fix your submission before you submit - do not submit an incomplete module! You can receive personal help and/or a short extension if you contact a supervisor.

Remember to confirm your successful self-check!