Functions and Equations

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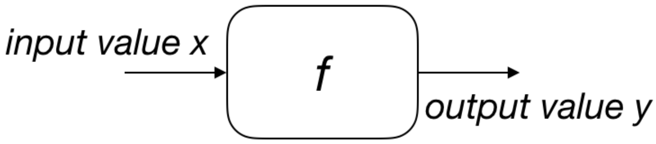
1 Functions



Definition (Function)**.** Let A and B be sets. A *function* f from domain A to co-domain B, written $f : A \to B$ is a mapping that relates to each element x in A a single element in B, denoted $f(x)$.

Example. *Let $\mathbb{N}$ be the set of natural numbers (i.e. non-negative integers). The* successor function *is the function* $succ\colon \mathbb{N} \to \mathbb{N}$ *given by* $succ(n) = n + 1$ *for every natural number* n*.*

It is convention to use letters f, g and h as names for functions, but one is free to assign any name, such as *succ* above, or $myfunction$, $average$, etc. Also, most times we don't specify the domain and co-domain of a function if it is obvious what these are.

Example.

1. $f(x) = 2x$;
2. $g(x) = ax + b$ *for some fixed* a *and* b;
3. $h(x) = \sqrt{x}$.

Notice that f and g can be treated as functions with domain any set of real numbers. However, h is only defined for $x \geq 0$, so the natural choice of domain for h is the non-negative real numbers.

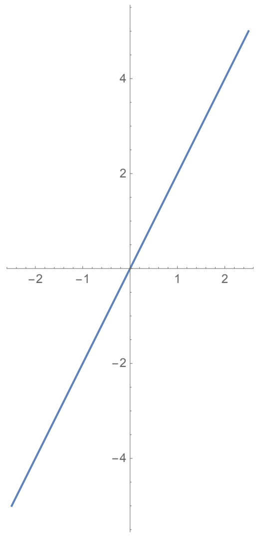
Evaluating functions. Let f and g be as in the previous Example. Then we have

$$f(0) = 0 \qquad g(0) = b$$
$$f(2) = 4 \qquad g(1) = a + b$$
$$f(-5) = -10 \qquad g(-\tfrac{b}{a}) = 0 \text{ (provided } a \neq 0)$$

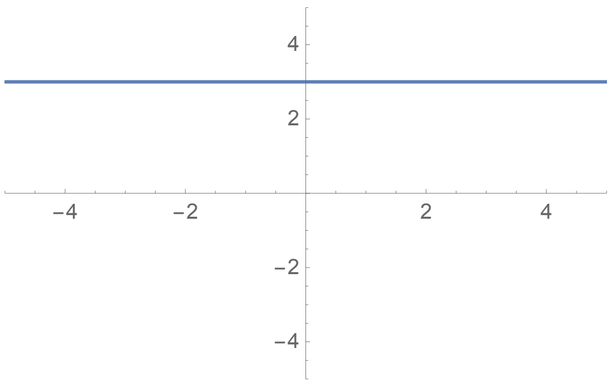
Functions can be illustrated by plotting $f(x)$ against x:

Example. *Some examples of plotting functions.*

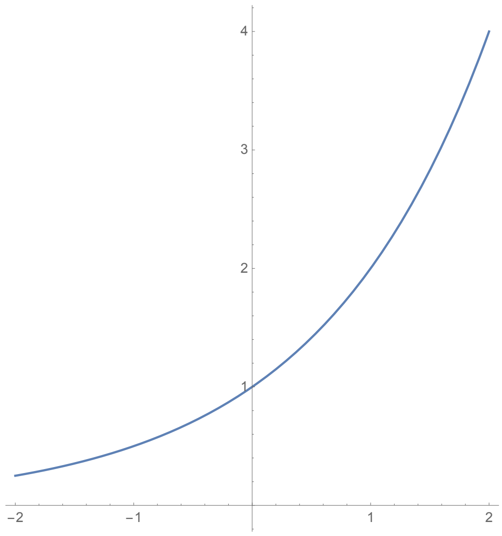
- $f(x) = 2x$ *(doubling function):*



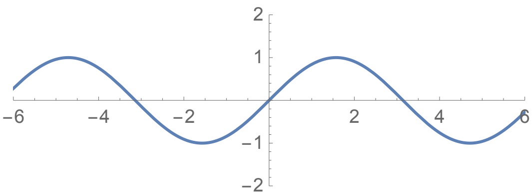
- $f(x) = 3$ *(constant function):*



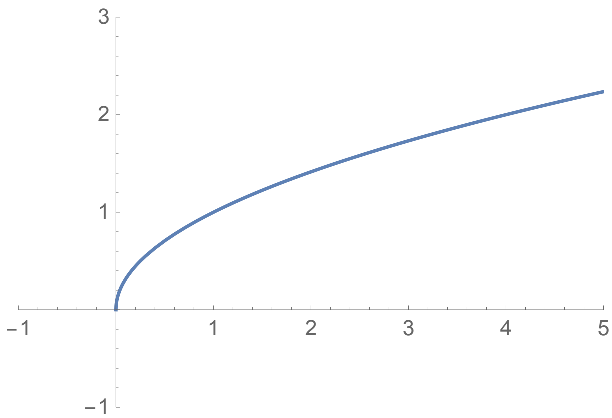
- $f(x) = 2^x$ *(exponential function):*



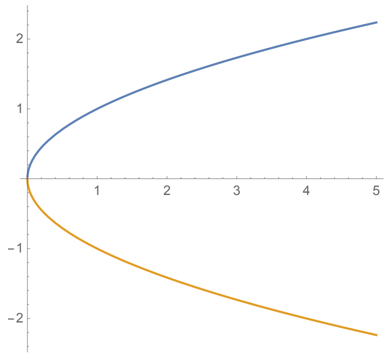
- $f(x) = \sin(x)$ *(sine function):*

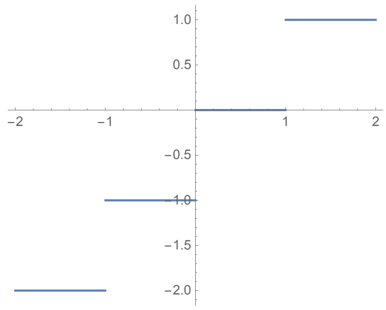


- $f(x) = \sqrt{x}$ *(square-root function):*

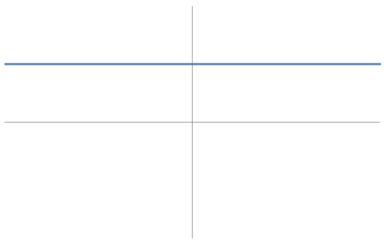
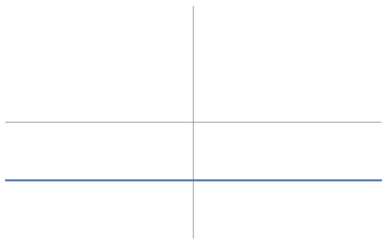


Question. *Do the following graphs represent functions?*

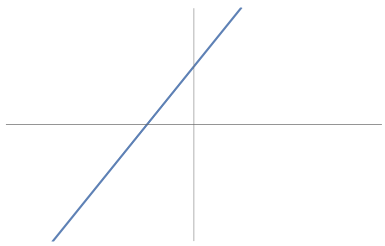




Example. *We will plot the function* $g(x) = ax + b$*. Since we do not know the values of* a *and* b *we can only determine the shape of the graph. For instance, if* $a = 0$ *we have* $g(x) = b$*, a constant function, so depending on whether* b *is positive, negative we obtain one of the following two graphs.*

If $a \neq 0$ *then* $g(-\frac{b}{a}) = 0$ *and, since* $g(0) = b$*,* g *describes a straight line through the coordinates* $(-\frac{b}{a}, 0)$ *and* $(0, b)$*. So, depending on the sign of* a *and* b*, there are four possibilities. For example, if* $a, b > 0$ *the graph of* g *looks like*



1.1 Proportionality

Given two functions f and g, we say that f is *proportional* to g if $\frac{f(x)}{g(x)}$ is constant, in other words, if $f(x) = kg(x)$ for some fixed $k \neq 0$. We call k the *proportionality constant* and that f and g are proportional is denoted by $f \propto g$.

Example.

- *Force under gravity* $\propto$ *mass, with proportionality constant* g *– equation:* $F(m) = mg$*.*

- *Area of circle* $\propto$ *radius*^{2}*, with proportionality constant* π *– equation:* $A(r) = \pi r^2$.

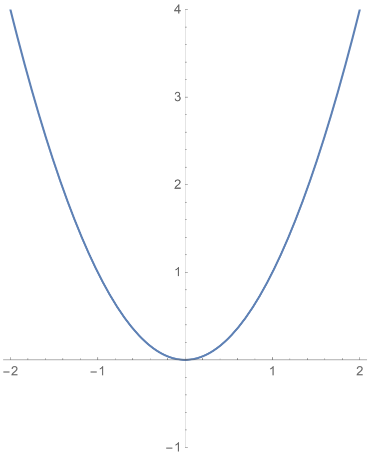
1.2 Linear functions

Linear functions have the form $f(x) = ax + b$ for some fixed a and b. Notice that if $b = 0$ then $\frac{f(x)}{x} = a$, so x is proportional to the identity function $id(x) = x$.

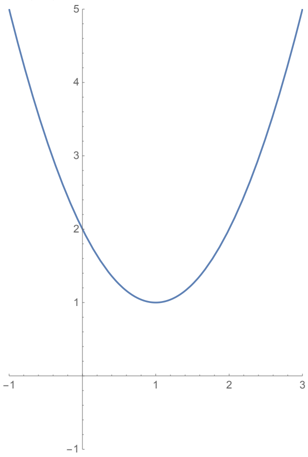
1.3 Quadratic functions

General form: $f(x) = ax^2 + bx + c$ for some fixed a, b and c.

Example.



- $f(x) = x^2$ *has the graph*
- $f(x) = x^2 - 2x + 2$. *This is* $f(x) = (x-1)^2 + 1$ *and has the graph:*

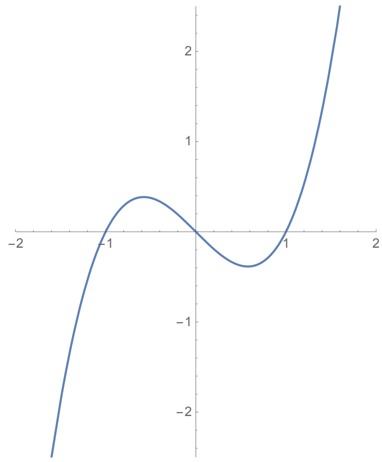


Question. *Plot the function* $f(x) = 2x - x^2$.

1.4 Polynomials

General form: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$. The *degree* of a polynomial of the form above is the largest n such that $a_n \neq 0$. Linear and quadratic functions are examples of polynomials of degree 1 and 2 respectively. Polynomials of degree 3 are called *cubic*.

Example. $f(x) = x^3 - x$. *As* $x^3 - x$ *can be written as* $x(x-1)(x+1)$ *we see that there are three values of* x *such that* $f(x) = 0$*:* $x = -1, 0, 1$. *Below is the graph of* f.

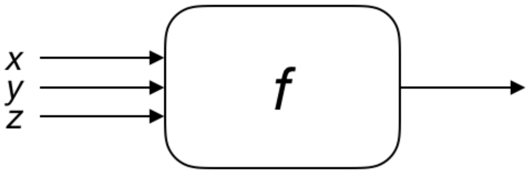


1.5 More functions

Look at any basic mathematics textbook for examples of:

- exponentials
- logarithms
- trigonometric functions
- reciprocals
- factorial

1.6 Functions on multiple inputs



Example.

1. *mult*$(x, y) = x \times y$ – *multiplication function*
2. $F(m_1, m_2, r) = G\frac{m_1 m_2}{r^2}$ – *gravitational force*
3. *average*$(x_1, x_2, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n}$ – *average function.*

1.7 Linear and quadratic multivariate functions

General form of linear functions in n variables:

$$l(x_1, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b$$

General form of quadratic functions in 2 variables:

$$q(x, y) = ax^2 + by^2 + cxy + dx + ey + f$$

2 Equations

An equation is a statement expressing equality between two quantities.

Example. *The following are all equations:*

i) $x = 1$

ii) $4x = 52$

iii) $3x = 2y$

iv) $x^2 + y^2 = 25$

v) $ax + b = 0$

vi) $x^2 + x = 2$

vii) $y^2 = x$

Here are some facts and properties about equations.

1. You can always rewrite an equation in the form of

$$f(x, y, z, \dots) = 0 \tag{1}$$

for some function f. For example, the equation $3x = 2y$ can be rewritten as $3x - 2y = 0$

2. An equation is called *linear* if, when written in the form of (1), the function f is a linear function. Quadratic and cubic equations are similarly defined. The equations (i), (ii), (iii) and (v) in the above example are linear, and the rest are quadratic.

3. A *solution* to an equation is an assignment of values to variables in the equation that make the equation true. For example, the assignment $x \mapsto 13$ makes the equation (ii) true and hence is a solution to that equation. The pair of assignments $\{x \mapsto 3, y \mapsto 4\}$ yields a solution to equation (iv).

4. Equations don't always have unique solutions. For example, equation (vi) has two solutions, namely $x \mapsto 1$ and $x \mapsto -2$. Equation (iii) has infinitely many solutions: for every real number r the pair of assignments $\{x \mapsto r, y \mapsto \frac{3r}{2}\}$ is a solution.

5. Equations should be thought as imposing constraints or restrictions on the possible values variables can take.

6. There are methods to solve certain kinds of equation. We will look at some of these next.

2.1 Solving equations

When we solve equations we make use of some basic principles. One such is that adding or subtracting the same quantity from both sides of an equation does not alter the solutions to the equation. We already saw this principle at work in the equation (1). Subtracting '$2y$' from both sides of equation (iii) yields the equation $3x - 2y = 2y - 2y$, that is, $3x - 2y = 0$. Any solution to this new equation is a solution to the original one, and vice versa.

Another principle is that multiply or dividing both sides of an equation by any *fixed, non-zero* number preserves solutions. For example, $x^2 + y^5 = 25$ has the same solutions as $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$.

2.1.1 Solving linear equations

From the equation $ax + b = 0$ we first obtain $ax = -b$ and then, provided $a \neq 0$, the equation $x = -\frac{b}{a}$ which clearly has as its only solution the assignment $x \mapsto -\frac{b}{a}$.

2.1.2 Solving quadratic equations

Consider the quadratic equation

$$ax^2 + bx + c = 0 \quad \text{where } a \neq 0.$$

Equations of the form above may have 0, 1 or 2 solutions depending on the values of a, b and c (you may skip forward to the summary box to see the general solutions). To find the solutions we can use the following method, known as *completing the squares*.

1. First divide both sides by a to get $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.
2. Subtract $\frac{c}{a}$ from both sides: $x^2 + \frac{b}{a}x = -\frac{c}{a}$.
3. Now add $\frac{b^2}{4a^2}$ to both sides: $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$.
4. Now we can rewrite the lefthand side as $(x + \frac{b}{a})^2$ and simplify the righthand side to get $(x + \frac{b}{a})^2 = \frac{b^2-4ac}{4a^2}$.
5. If the quantity on the righthand side is positive, we can take the square root, yielding two equations: $x + \frac{b}{a} = \sqrt{\frac{b^2-4ac}{4a^2}}$ and $x + \frac{b}{a} = -\sqrt{\frac{b^2-4ac}{4a^2}}$.
6. Solving the two linear equations above (and simplifying further) we get two solutions

$$x \mapsto \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad\qquad x \mapsto \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{2}$$

To summarise,

- If $b^2 > 4ac$, the equation has exactly two solutions, given in (2).
- If $b^2 = 4ac$, then the equation has a single solution $x \mapsto \frac{-b}{2a}$.
- If $b^2 < 4ac$, the quadratic equation has no real solutions.

2.1.3 Solving cubic, polynomial and more complicated equations

There are methods that apply to certain kinds of equation, but no single method applies to all types.

2.2 Systems of equations

Often we are presented with multiple equations and asked whether we can find an assignment that is a solution to all the equations.

Example. *Let S be the system of equations*

$$S : \begin{cases} y = 3x + 1 \\ 2y = 10 - 2x \end{cases}$$

which consists of two linear equations. Each equation, considered in isolation, has infinitely many solutions. But which of these solutions (and how many of them) are solutions to both equations simultaneously?

A general method to solve systems of linear equations is to eliminate the unknown variables one by one. The main principle we use is that adding or subtracting one equation from another (see below) does not alter solutions.

Example. *To solve the system S above, we start by multiplying the first equation by -2 to obtain the equations*

$$\begin{cases} -2y = -6x - 2 \\ 2y = 10 - 2x \end{cases}$$

Now adding the two sides of the equations together, we get the single equation $-2y + 2y = (-6x - 2) + (10 - 2x)$, which simplifies to $0 = 8 - 8x$. Notice that we have eliminated the variable y, so this latter equation can be solved to find $x \mapsto 1$. Plugging in 1 for x in either of the starting equations gives $y \mapsto 4$. So the simultaneous solution to the system S is $\{x \mapsto 1, y \mapsto 4\}$.

Example. *Let S' be system of three equations in three unknowns given by*

$$S' : \begin{cases} x + y + z = 5 \\ 2x + 3y - z = -1 \\ -x + y + 2z = 9 \end{cases}$$

Adding together the first and second equation we get $3x + 4y = 4$, and adding twice the second equation to the third we get $3x + 7y = 7$, yields two equations in which the unknown z has been eliminated. Now we can consider the system

$$\begin{cases} 3x + 4y = 4 \\ 3x + 7y = 7 \end{cases}$$

of two equations in two unknowns, which we know how to solve. Once we have obtained the solution $\{x \mapsto 0, y \mapsto 1\}$ we can use any of the original equations in S' to determine a value for z.

There are a number of algorithms to solve systems of linear equations in any number of unknowns. If you are interested, we recommend reading about *Gaussian Elimination* which is, essentially, what we were using in our two examples above.

Systems of linear equations do not always have a unique solution. Consider the two systems

$$\begin{cases} y = 3x + 1 \\ 2y = 10 - 2x \\ x = -y \end{cases} \qquad \begin{cases} x + y + z = 5 \\ 2x + 3y - z = -1 \end{cases}$$

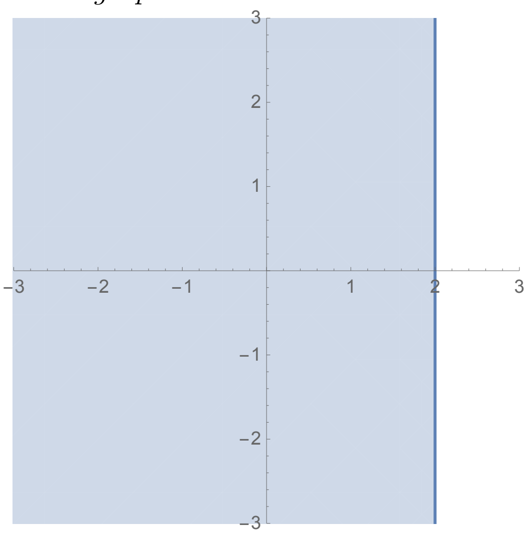
One has no solutions and one has infinitely many.

3 Inequalities

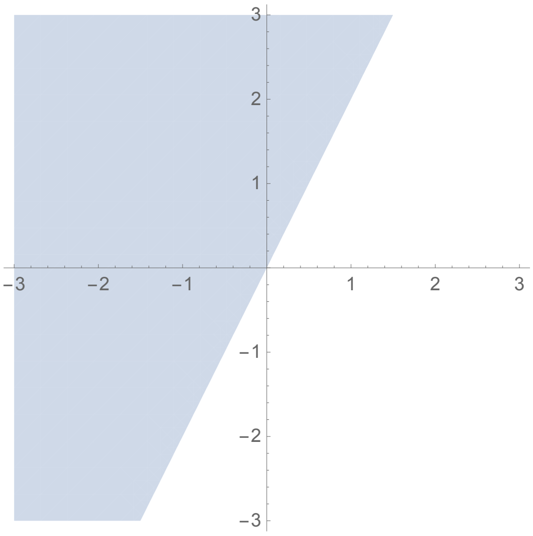
An *inequality* is a statement expressing one of the relations 'less than', 'greater than', 'less than or equal', or 'greater than or equal' between two quantities.

Example.

- $x \leq 2$ *describes the shaded region (all points on and 'left' of the line* $x = 2$*) in the graph*



- *The inequality* $y > 2x$ *describes the shaded area in the graph below. Note that the points on the line* $y = 2x$ *are excluded.*



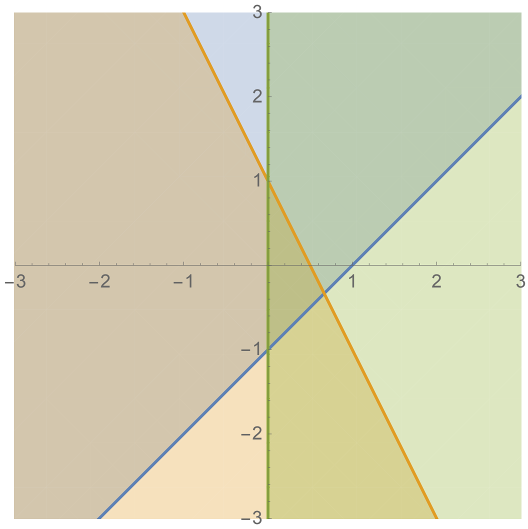
You can always rewrite inequalities by adding, subtracting etc by the same value on both sides but be careful when multiplying or dividing by negative values: $p \leq q$ if, and only if, $-p \geq -q$.

3.1 Systems of inequalities

Example.

$$S : \begin{cases} y \geq x - 1 \\ y \leq 1 - 2x \\ x \geq 0 \end{cases}$$

Above we have a system of three inequalities, whose simultaneous solutions are the points of the shaded triangle formed by the intersection of the three lines in the figure: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