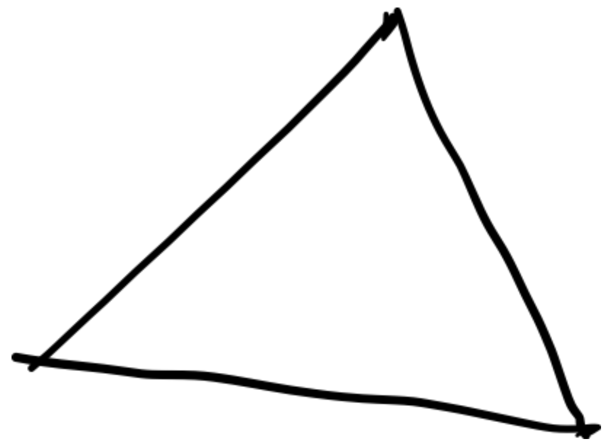
Introduction to mathematical thinking

Dag Wedelin

Imagine a triangle!

Is this a triangle?

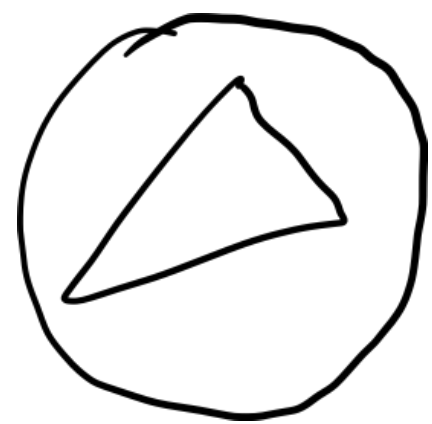


Is there any difference between this picture and the triangle you imagine in your mind when you see it?

Imagine a triangle inside a circle.

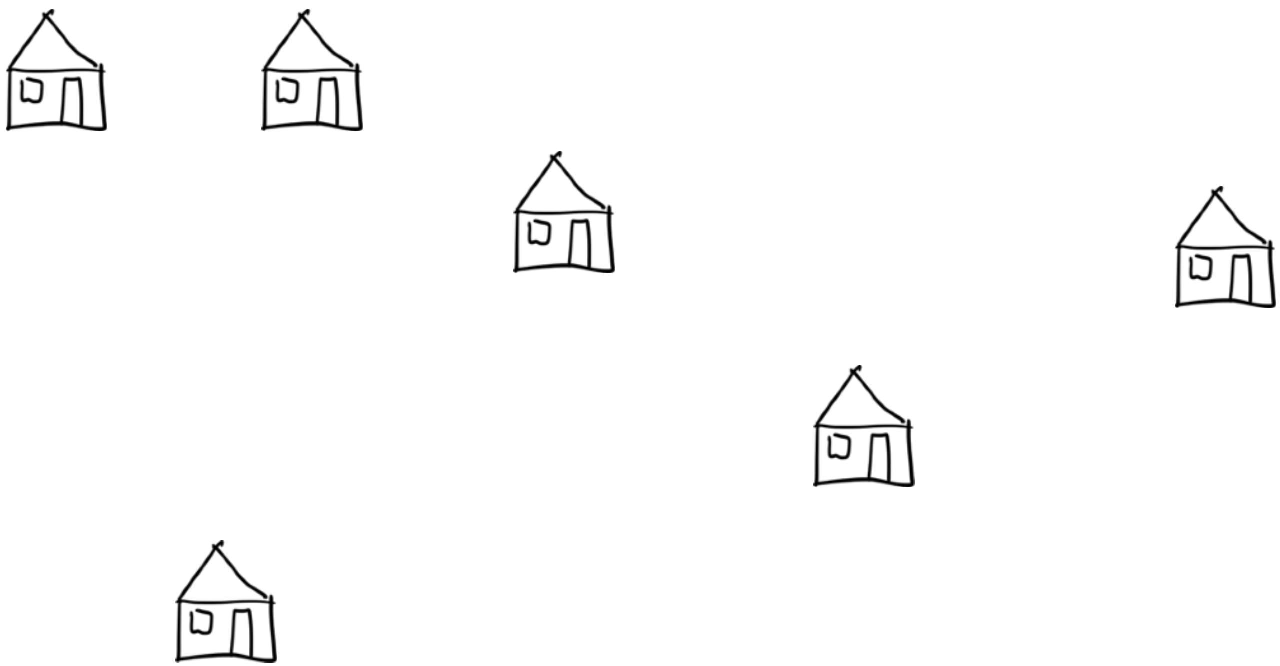
Imagine that you can move the corners of the triangle.

What is the largest triangle you can have in the circle?

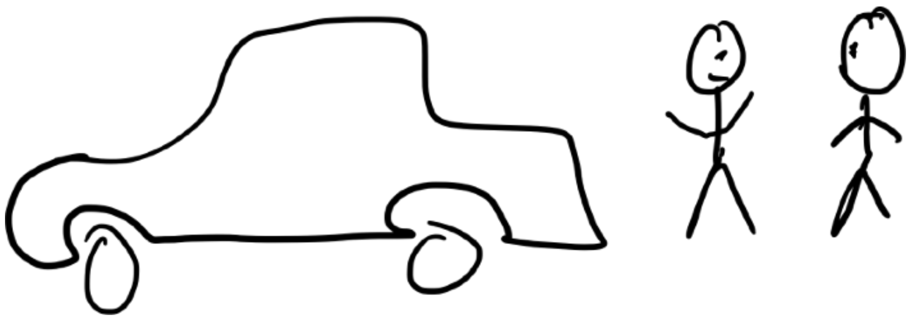


We are naturally able to imagine and reason about abstract mathematical concepts!

Say that we want to connect some houses to the electric grid, using the shortest possible cabling.



Would the connections have any particular structural property?



- How much fuel do we have?
- Quite a lot
- So how far can we go?
- Pretty far
- Will it be sufficient for our trip?
- ...

What would we need to draw a meaningful conclusion?

Imagine seven overlapping circles
and the areas between them!

Imagine seven overlapping circles
and the areas between them!

Remember the number
746859128056447838496462337012753

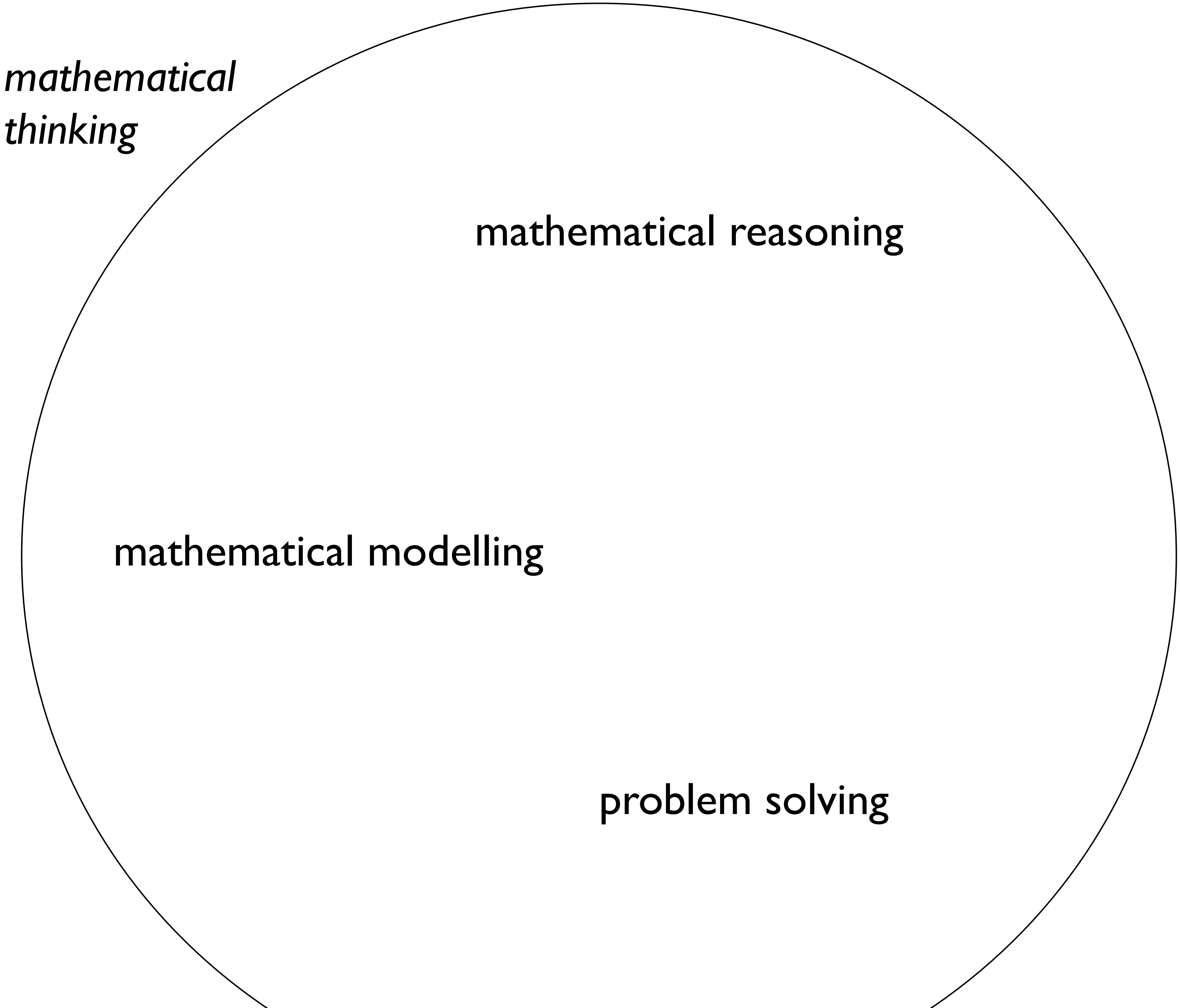
mathematical thinking

Precision is important – ambiguity makes it difficult to draw conclusions

Some ways to reason provide certain conclusions, others just suggest possibilities.

We can relate aspects of real situations to abstract patterns.

When it gets complicated, we quickly reach our limits!


mathematical
thinking
mathematical reasoning
mathematical modelling
problem solving

mathematical reasoning

Mathematical reasoning is about <u>creating a well-defined situation and drawing conclusions in small steps</u> as far as you can

Also within mathematics we have intuitive questions!

Is infinity a number?

What is a function?

What is randomness?

$A \longrightarrow B$

premise conclusion

Deductive (certain) reasoning
lets us *deduce* new true statements from old ones

If $a<b$ and $b<c$ then $a<c$

We have a triangle in a circle. If the triangle has a corner inside the circle, it can be made larger by moving the corner towards the edge => the largest triangle must have all corners on the edge

A chain of deductive arguments is called a derivation, a derivation of a given statement is called a proof

Calculations are just a particular kind of derivation!

$11 \cdot 44 =$

$10 \cdot 44 + 44 =$

$440 + 44 =$

$\underline{484}$

a derivation

Theorem $11 \cdot 44 = 484$

Proof

$$\begin{array}{rr} & 11 \\ \times & 44 \\ \hline & 44 \\ & 44 \\ \hline & 484 \end{array}$$

QED

a derivation formatted as a theorem and a proof

Pure mathematics is concerned with whether certain kinds of precise statements are true of false

There are infinitely many prime numbers

For every real number a, the equation x^2+a=0 has a real root

The square root of 2 is not a rational number

If p(n) denotes the number of primes less than or equal to the natural number n, then as n becomes very large, p approches n/log n

The ultimate goal is therefore to explain all statements deductively!

However, discovering and proving interesting results, require extensive investigation, exploration and creativity, <u>using all forms of reasoning</u>.

Plausible (non-certain) reasoning
lets us draw tentative conclusions (which may not be true but can be investigated further)

We have tested and found that:
7+3 = 3+7
11+5 = 5+11
2+14 = 14+2
=> **<u>maybe</u>** the order never matters!

(inductive reasoning – generalizing from instances)

If it had been A then we would have seen footsteps in the garden, but we don't.
=> so this **<u>supports</u>** that it might be B

(abductive reasoning – finding the best explanation)

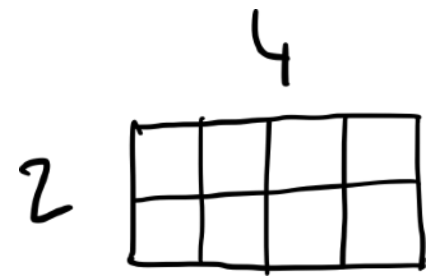
there are several signs…
maybe I have seen this before… so I have a feeling that…
⇒ this **<u>might be</u>** way to the lake…

("intuitive" reasoning)

Plausible reasoning leads us to a hypothesis or conjecture

Method of conjecture and proof

$$26 \times 31 = 806$$
$$31 \times 26 = 806$$

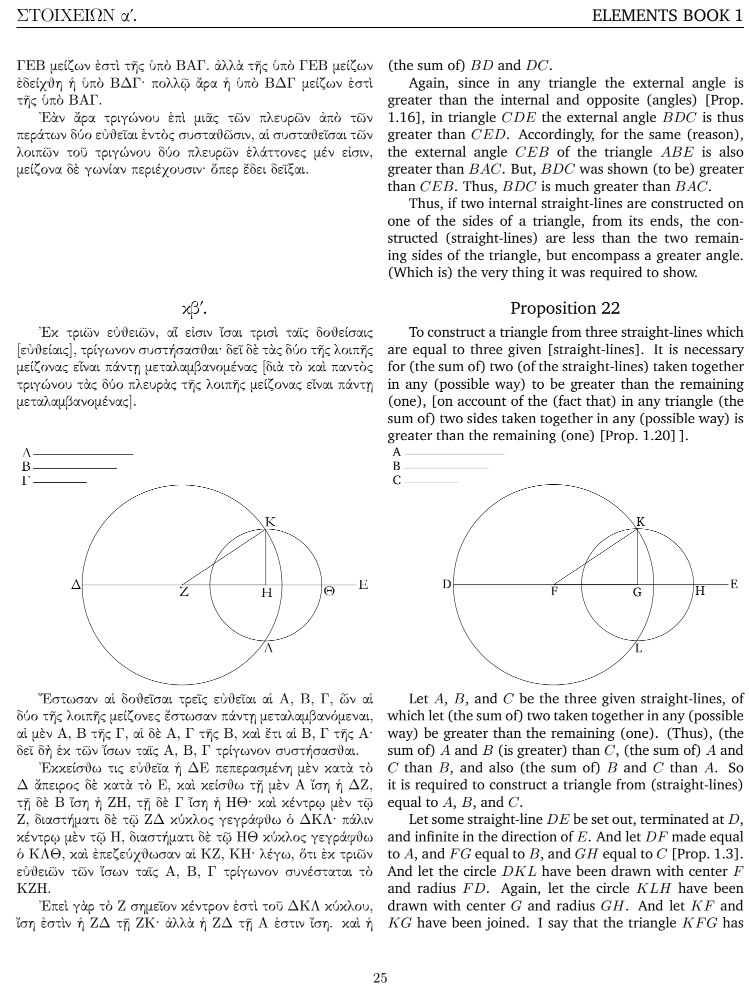


conjecture

proof

In mathematics, we usually first experiment, investigate and guess

Only thereafter we prove



ΣΤΟΙΧΕΙΩΝ α΄. ELEMENTS BOOK 1

ΓΕΒ μείζων ἐστὶ τῆς ὑπὸ ΒΑΓ. ἀλλὰ τῆς ὑπὸ ΓΕΒ μείζων ἐδείχθη ἡ ὑπὸ ΒΔΓ· πολλῷ ἄρα ἡ ὑπὸ ΒΔΓ μείζων ἐστὶ τῆς ὑπὸ ΒΑΓ.

Ἐὰν ἄρα τριγώνου ἐπὶ μιᾶς τῶν πλευρῶν ἀπὸ τῶν περάτων δύο εὐθεῖαι ἐντὸς συσταθῶσιν, αἱ συσταθεῖσαι τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ἐλάττονες μὲν εἰσιν, μείζονα δὲ γωνίαν περιέχουσιν· ὅπερ ἔδει δεῖξαι.

(the sum of) BD and DC.

Again, since in any triangle the external angle is greater than the internal and opposite (angles) [Prop. 1.16], in triangle CDE the external angle BDC is thus greater than CED. Accordingly, for the same (reason), the external angle CEB of the triangle ABE is also greater than BAC. But, BDC was shown (to be) greater than CEB. Thus, BDC is much greater than BAC.

Thus, if two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) are less than the two remaining sides of the triangle, but encompass a greater angle. (Which is) the very thing it was required to show.

κβ΄.

Ἐκ τριῶν εὐθειῶν, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις [εὐθείαις], τρίγωνον συστήσασθαι· δεῖ δὲ τὰς δύο τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας [διὰ τὸ καὶ παντὸς τριγώνου τὰς δύο πλευρὰς τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας].

Proposition 22

To construct a triangle from three straight-lines which are equal to three given [straight-lines]. It is necessary for (the sum of) two (of the straight-lines) taken together in any (possible way) to be greater than the remaining (one), [on account of the (fact that) in any triangle (the sum of) two sides taken together in any (possible way) is greater than the remaining (one) [Prop. 1.20]].

A, B, Γ, Δ, Z, H, Θ, E, K, Λ

A, B, C, D, F, G, H, E, K, L

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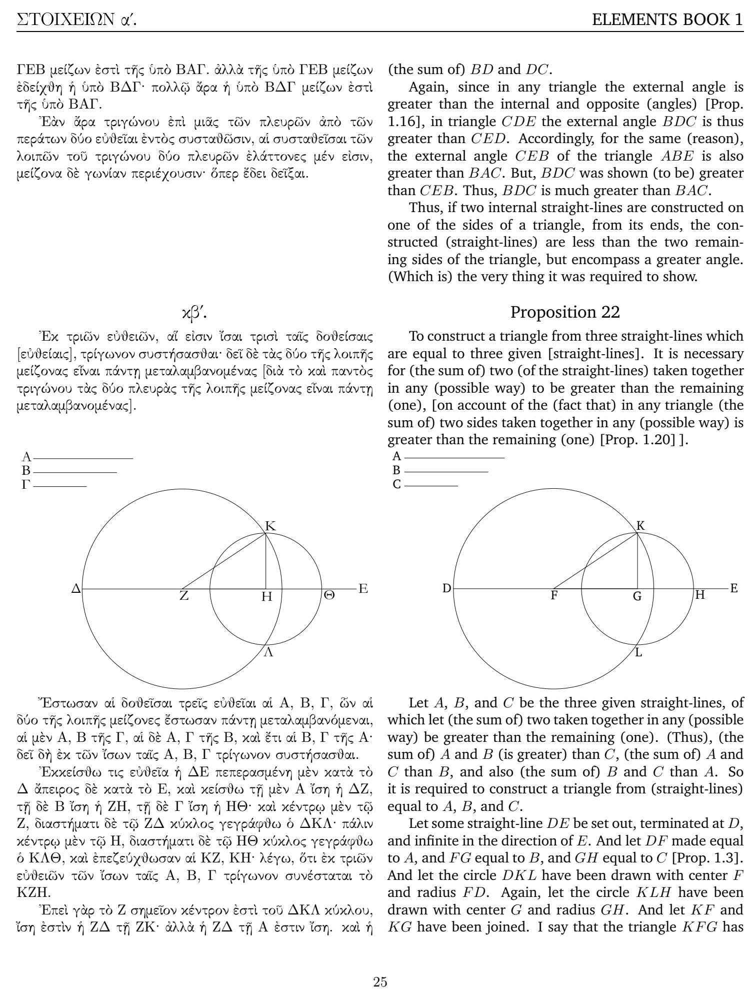
25

Thinking creates knowledge!



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The story of how people create mathematics is usually not told!

We must be careful in our reasoning!

"All pirates are cruel, and since you are cruel you must be a pirate!"

I took the medicine and I was well two days later => the medicine worked!

…

How to successfully and correctly reason is central in all mathematical thinkning.

The same conclusion can usually be reached in different ways

How to reason is sometimes more important than the result!

What is the relationship between reasoning and explaining?

Many people think that mathematics must be explained in some particular format. This is wrong! It is the reasoning that must be sound.

A formula can be good - but sometimes a figure and a text is much easier.

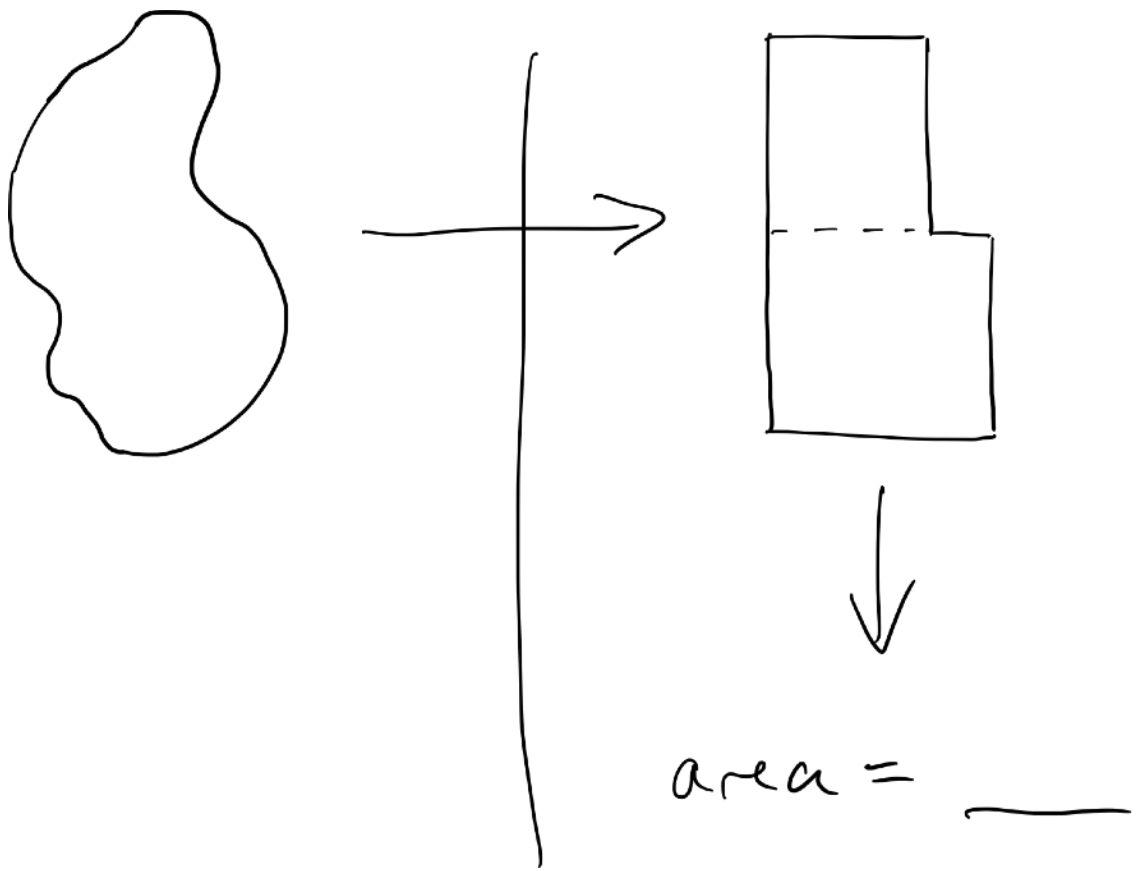
However, always take care so that statements are unambiguous!

So explain as you like!

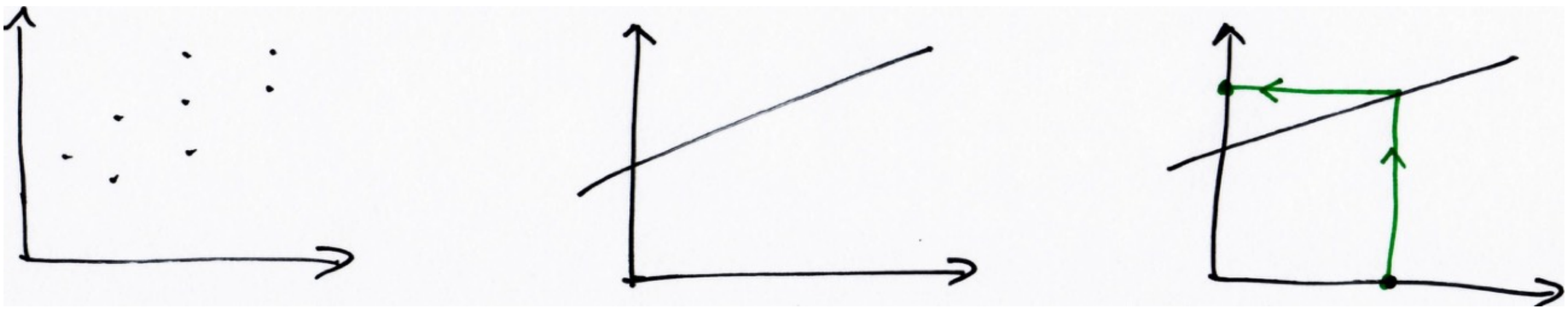
mathematical modelling

Why models?

"a convenient way to represent reality so that we more easily can draw conclusions about it"



Mathematical modelling and mathematical reasoning

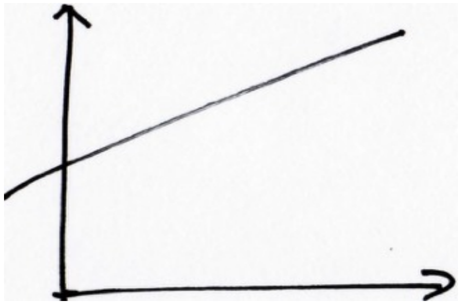


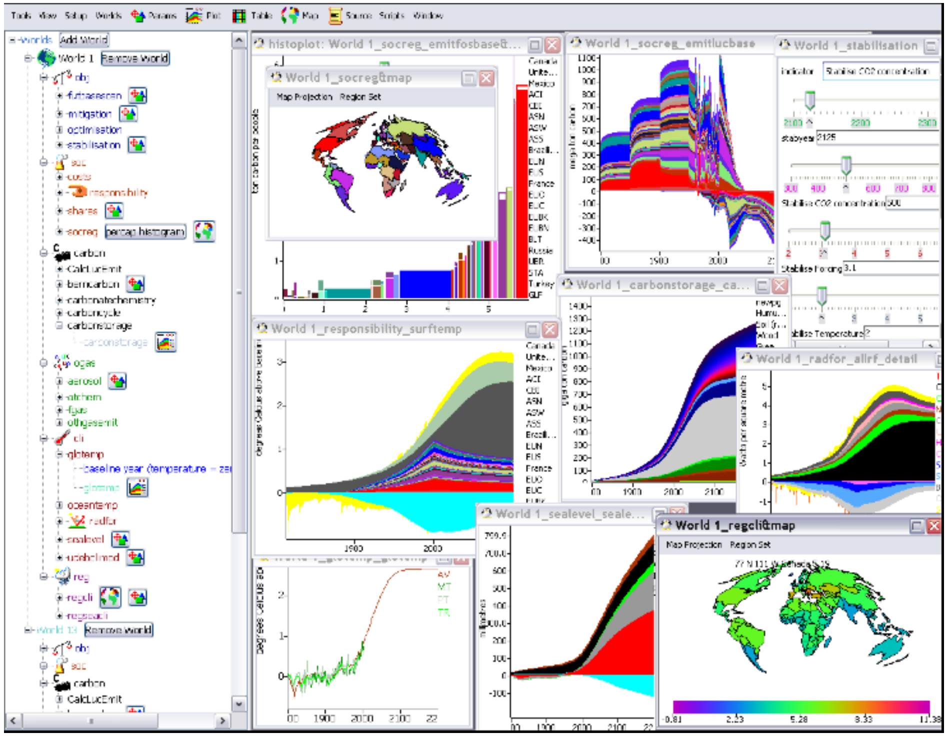
some aspect of reality ⟶ mathematical model ⟶ conclusions from model

modelling
often intuitive and inductive

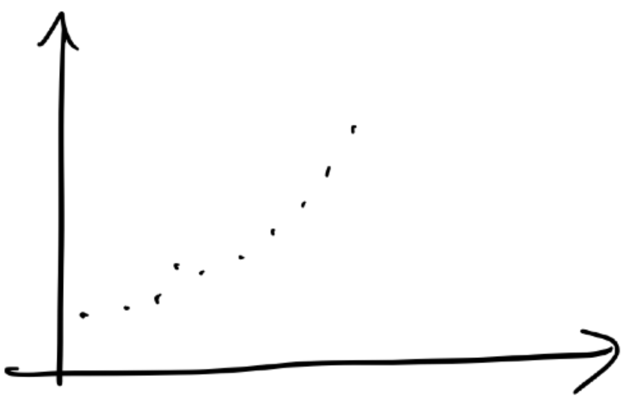
mathematical reasoning
often deductive

Simple and complicated models





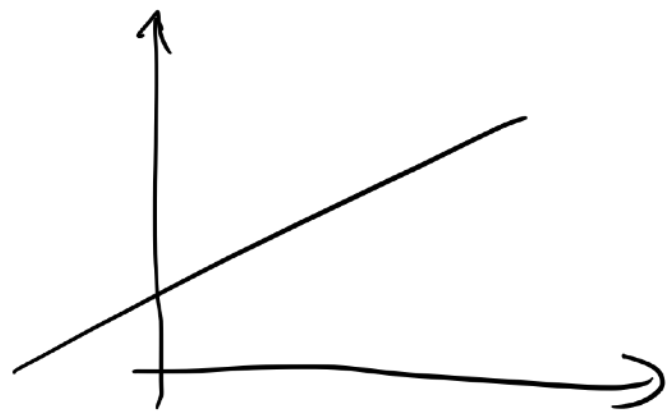
Example population growth



our data

parameters

$$p = a\,t + b$$



try something simple!

Can we improve?

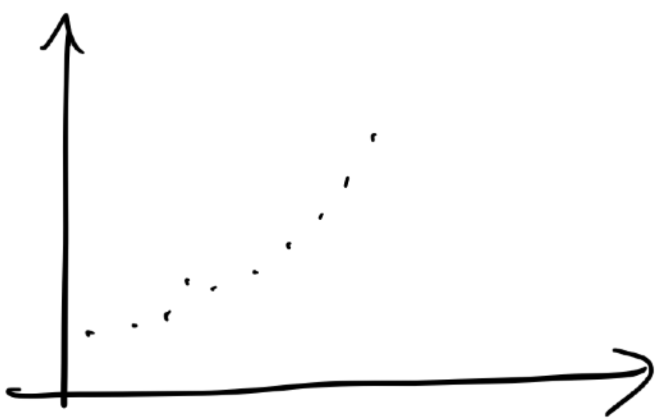
think!

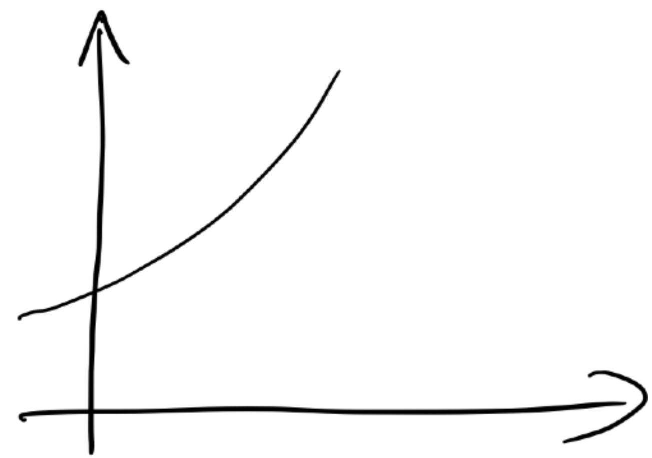
mechanistic
modelling
(deduction)
(or mathematical simplicity!)

look at the
data!

empirical
modelling
(induction)

$$p = c \cdot e^{at}$$





better!

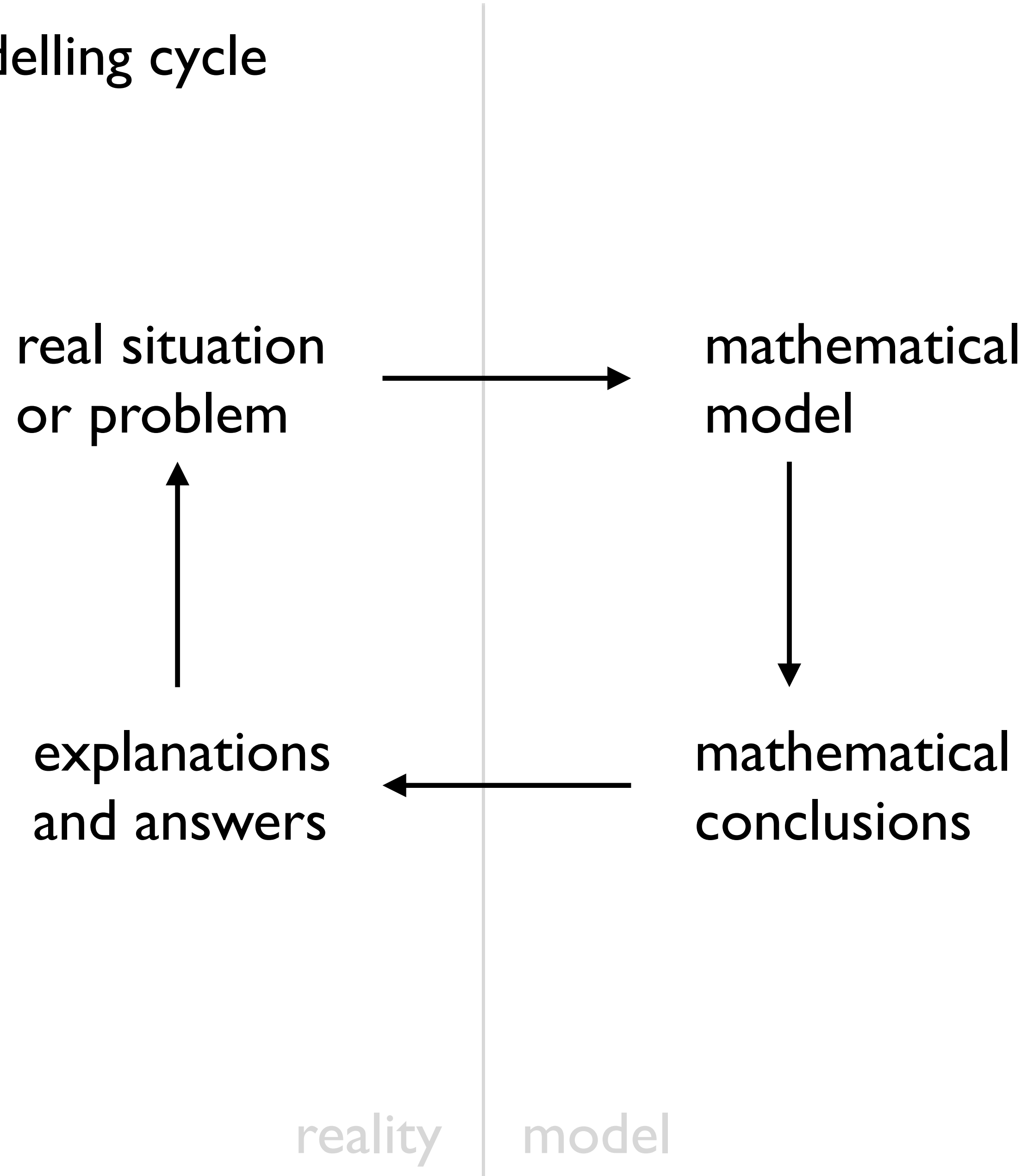
$$p(t) = \frac{a}{1 + c e^{-dt}}$$

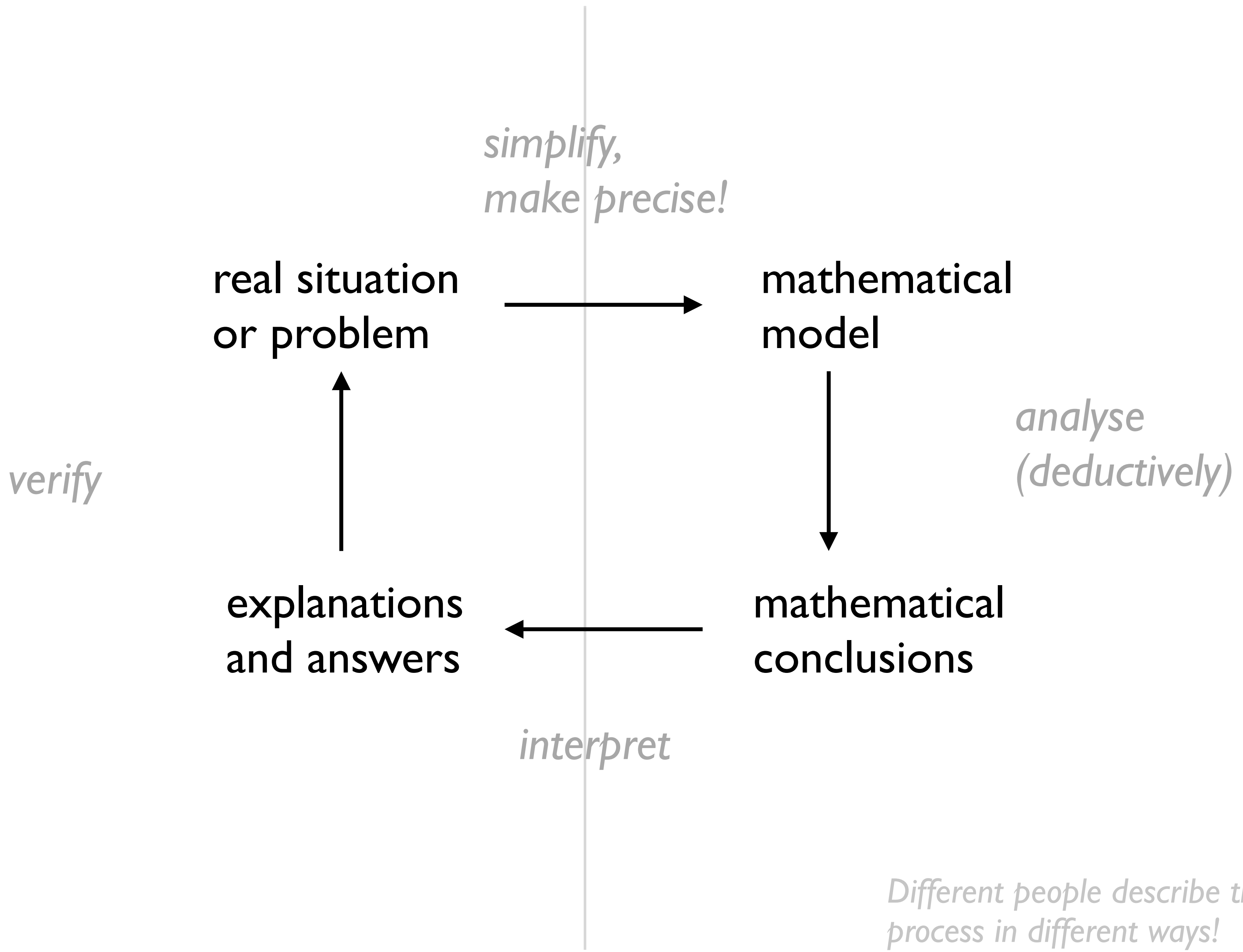
even better!

but also more complicated

we will discuss later how to find this function, but intelligent guessing is possible

The modelling cycle

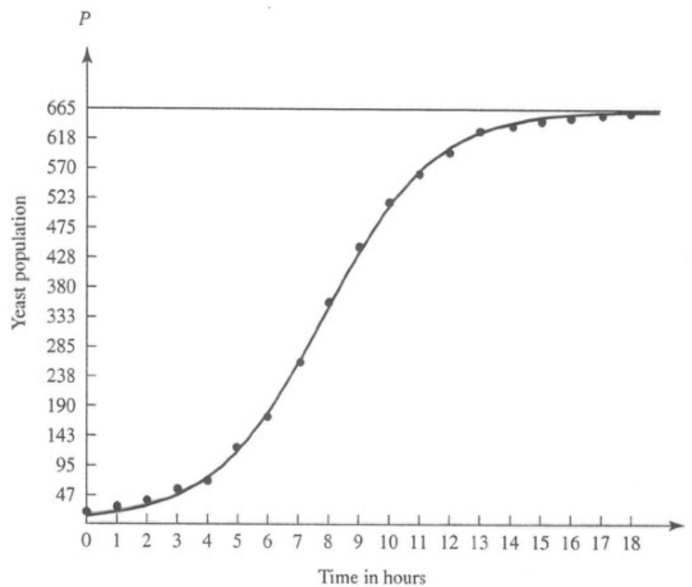




Different people describe this
process in different ways!

Verification of the model

Figure 10.6
Logistic curve showing the growth of yeast in a culture based on the data from Table 10.1 and the Model (10.13); the small dots indicate the observed values



So models help you to understand, explain, predict, ...

Table 10.2 Population of the United States from 1790 to 2000, with predictions from Equation (10.14)

Year	Observed population	Predicted population	Percent error
1790	3,929,000	3,929,000	0.0
1800	5,308,000	5,336,000	0.5
1810	7,240,000	7,227,000	−0.2
1820	9,638,000	9,756,000	1.2
1830	12,866,000	13,108,000	1.9
1840	17,069,000	17,505,000	2.6
1850	23,192,000	23,191,000	−0.0
1860	31,443,000	30,410,000	−3.3
1870	38,558,000	39,370,000	2.1
1880	50,156,000	50,175,000	0.0
1890	62,948,000	62,767,000	−0.3
1900	75,995,000	76,867,000	1.1
1910	91,972,000	91,970,000	−0.0
1920	105,711,000	107,393,000	1.6
1930	122,755,000	122,396,000	−0.3
1940	131,669,000	136,317,000	3.5
1950	150,697,000	148,677,000	−1.3
1960	179,323,000	159,230,000	−11.2
1970	203,212,000	167,943,000	−17.4
1980	226,505,000	174,941,000	−22.8
1990	248,710,000	180,440,000	−27.5
2000	281,416,000	184,677,000	−34.4

The simplicity and precision demanded by mathematics, as well as the iterative development process, often contributes to a successively improved understanding of the studied real system.

A simple model helps us to see that something new happens here!

Modelling can be seen as *the first step of mathematical problem solving* for applied problems.

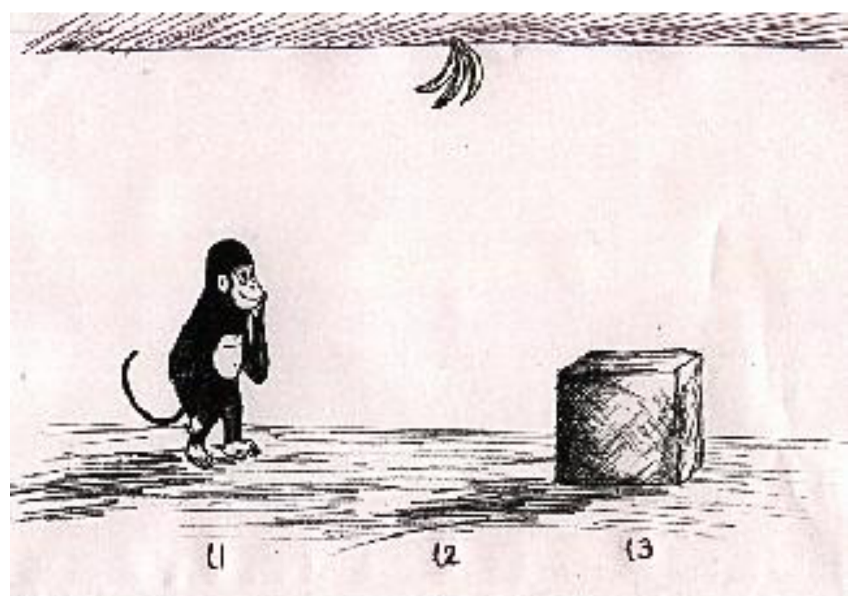
By creating a mathematical model you gain access to a vast range of mathematical and computational techniques.

Applied problems are not mathematical from the beginning. You must identify and formulate them as such - only then will you find use for the mathematics you know

Computer models bring life to your data!

problem solving

How can we make the most of our limited capacity?



Work in small manageable steps - (more power through many steps that you can actually implement)

Use effective tools - (more power in every step)

Use effective techniques - (more power by well chosen steps)

similarities with computing?

significant differences?

The notions of subproblems and subroutines become important, since they help you to break problems in smaller parts and also allow you to consider one thing at a time.

And as always when larger tasks and steps are involved, you must be able to think also on a strategic level, and not just about the details.

You then also need to *manage* your work and your time!

Typical workflow - easy or familiar problems

1. You easily understand the problem

2. You quickly see how to solve it

3. The problem is solved / implement the solution with no surprises

Typical workflow - intermediate problems

1. Understand the problem

2. Make a plan

3. Carry out the plan

4. Look back (check your result,
reflect on the process, ...)

(Polya)

Typical workflow – more difficult problems

1. Understand the problem
2. Make a plan
3. Carry out the plan
4. Look back

investigate for deeper understanding, define clearly

explore different approaches, begin with something simple!

when you fail you learn and go back

Continuously reflect, go back and revise, manage your time

This requires a lot of self-awareness!

You will have to struggle!

Never stop! Always do something!

You can always try to understand the problem better… Draw a figure, try small examples… Try to intuitively understand the problem and its solution…

Try to solve a simpler problem in a simple way…

Investigate extreme cases for easy thinking

If we expect our task to be a process in many steps, it becomes natural to engage in different subtasks that may shed light on the problem. Anything can be a clue!

investigate!

explore!

try things out -
different things

<u>be careful!</u>

manage your
time

Dare to fail – you will find out. The tenth time you may get it right.

(anonymous student)

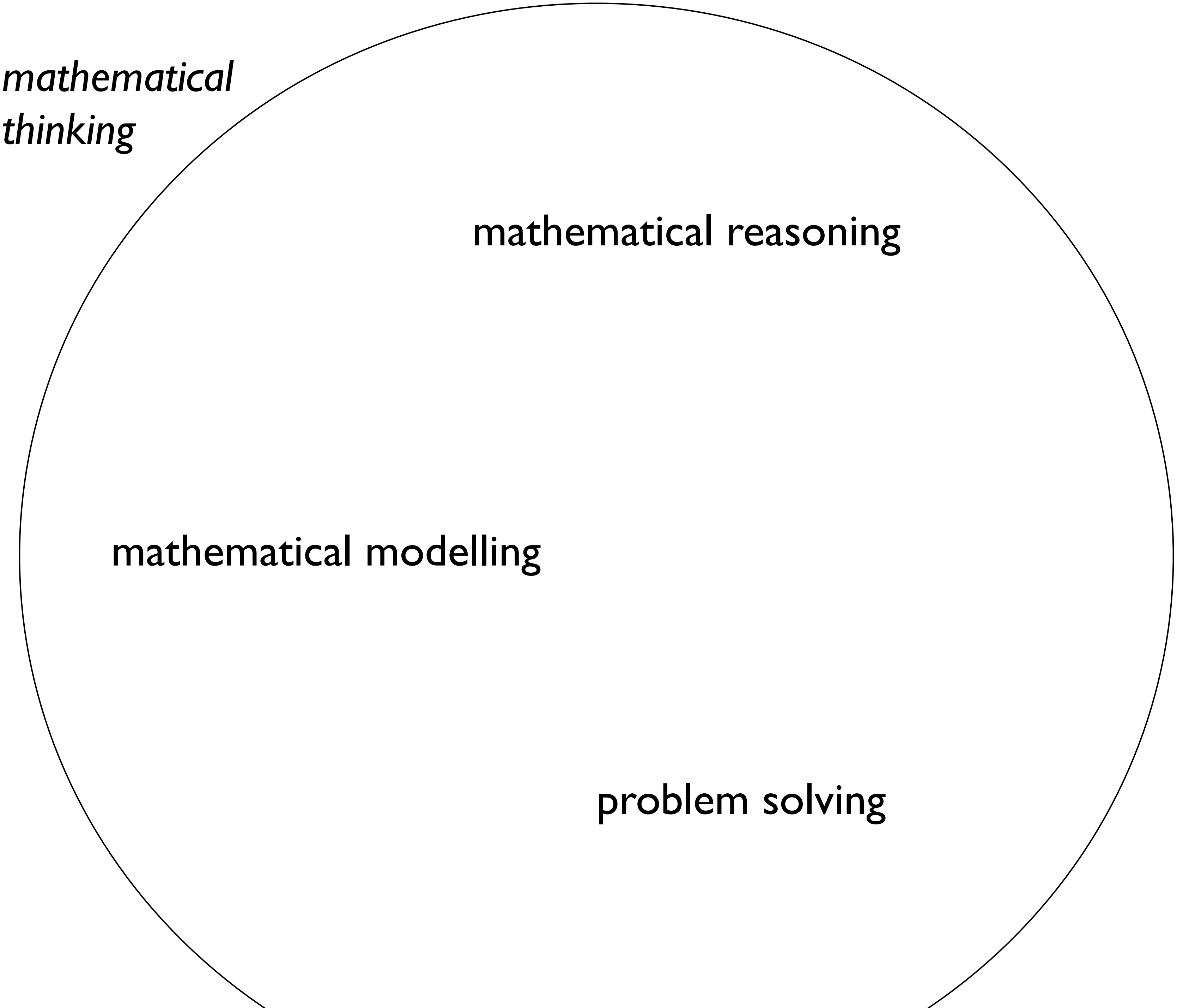
What is needed to solve a problem?

very different balance
for different problems

knowledge needed for solving a problem	**=**	**knowledge created by own thinking**	**+**	**knowledge from others**

because of the variation
you often have to add
something here!

mathematical thinking



mathematical thinking

IF p THEN q

$= \neg p \vee q$